

ON THE KNOT FLOER FILTRATION OF THE CONCORDANCE GROUP

STEPHEN HANCOCK, JENNIFER HOM, AND MICHAEL NEWMAN

ABSTRACT. The knot Floer complex together with the associated concordance invariant ε can be used to define a filtration on the smooth concordance group. We show that the indexing set of this filtration contains $\mathbb{N} \times \mathbb{Z}$ as an ordered subset.

1. INTRODUCTION

Two knots in S^3 are called *concordant* if they cobound a smooth, properly embedded cylinder in $S^3 \times [0, 1]$. The set of knots in S^3 , modulo concordance, forms an abelian group, the *concordance group*, denoted \mathcal{C} , where the operation is induced by connected sum. If a knot is concordant to the unknot, then we say that it is *slice*. The inverse of a knot K is given by $-K$, the reverse of the mirror of K . It is straightforward to show that K_1 and K_2 are concordant if and only if $K_1 \# -K_2$ is slice.

A powerful tool for understanding knots is the knot Floer complex, defined by Ozsváth and Szabó [OS04], and independently Rasmussen [Ras03]. To a knot K , they associate a $\mathbb{Z} \oplus \mathbb{Z}$ -filtered chain complex, denoted $CFK^\infty(K)$, whose filtered chain homotopy type is an invariant of K . Associated to the complex $CFK^\infty(K)$ is a $\{-1, 0, 1\}$ -valued concordance invariant $\varepsilon(K)$ defined in [Hom11]. The set of such filtered chain complexes forms a monoid under the operation of tensor product, and modulo an equivalence relation defined in terms of ε , this monoid can be made into a group, denoted \mathcal{F} .

The advantage of this approach is that there is a homomorphism from

$$\mathcal{C} \rightarrow \mathcal{F},$$

defined by $[K] \mapsto [CFK^\infty(K)]$. Moreover, the group \mathcal{F} has a rich algebraic structure coming from a total ordering. This ordering gives a filtration on \mathcal{F} that can be pulled back to a filtration on \mathcal{C} , called the *knot Floer filtration*. While the indexing set of the knot Floer filtration is largely unknown, our main theorem gives a lower bound on the complexity of this indexing set.

Theorem 1. *The indexing set of the knot Floer filtration contains a subset that is order isomorphic to $\mathbb{N} \times \mathbb{Z}$. Specifically, we can index the filtration by*

$$S = \{(i, j) \mid (i, j) \geq (0, 0)\},$$

where $S \subset \mathbb{Z} \times \mathbb{Z}$ inherits the lexicographical ordering. Furthermore, each successive quotient is infinite, i.e., for $(i, j), (i', j') \in S$ and $(i, j) < (i', j')$, we have that

$$\mathbb{Z} \subset \mathcal{F}_{(i', j')} / \mathcal{F}_{(i, j)}.$$

Filtrations have been shown to be an effective tool for studying the concordance group. For example, Cochran, Orr, and Teichner [COT03] define the n -solvable filtration

$$\cdots \subset \mathcal{F}_{-(n+1)} \subset \mathcal{F}_{-n.5} \subset \mathcal{F}_{-n} \subset \cdots \subset \mathcal{F}_{-1} \subset \mathcal{F}_{-0.5} \subset \mathcal{F}_0 \subset \mathcal{C},$$

a filtration indexed by negative half integers. (Note that we adopt the convention that an indexing set for a filtration \mathcal{F} is an ordered set S with the property that for $a, b \in S$, $a < b$ implies that

$\mathcal{F}_a \subset \mathcal{F}_b$.) It was shown by Cochran, Harvey, and Leidy [CHL09] that the quotient $\mathcal{F}_{-n}/\mathcal{F}_{-n.5}$ is of infinite rank for each non-negative integer n . Recent work of Cochran, Harvey, and Horn [CHH12] defines the bipolar filtration, again indexed by the negative natural numbers, and they also show that each successive quotient is of infinite rank.

Our approach to filtering the concordance group is in a certain sense dual to these earlier approaches. While the filtrations of [COT03] and [CHH12] probe the structure of the concordance group with a simple indexing set and large successive quotients, our approach employs a much more robust indexing set and smaller successive quotients.

The proof of our result requires the computation of a large family of knot Floer complexes, modulo “ ε -equivalence”. While computing the knot Floer complex in general is difficult, we use two properties of knot Floer homology that give us a large class of knots for which the computation simplifies drastically.

Recall that an L -space is a rational homology S^3 for which $\text{rk } \widehat{HF}(Y) = |H_1(Y; \mathbb{Z})|$, so named because this class of 3-manifold includes lens spaces. The first property that we use pertains to a family of knots called L -space knots, that is, knots which admit a positive L -space surgery. It is well-known that positive torus knots admit positive lens space surgeries and thus are L -space knots. Ozsváth and Szabó [OS05, Theorem 1.2] show that the knot Floer complex of an L -space knot K is completely determined by the Alexander polynomial of K . Moreover, Hedden [Hed09] proves that sufficiently large cables of L -space knots are again L -space knots. Thus, to understand the knot Floer complexes of torus knots and appropriate cables, it is sufficient to know the knot’s Alexander polynomial. It is well-known that the Alexander polynomial of $T_{p,q}$, the (p, q) -torus knot, is

$$\Delta_{T_{p,q}}(t) = \frac{(t^{pq} - 1)(t - 1)}{(t^p - 1)(t^q - 1)}$$

and that the Alexander polynomial of $K_{p,q}$, the (p, q) -cable of K , is

$$\Delta_{K_{p,q}}(t) = \Delta_K(t^p) \cdot \Delta_{T_{p,q}}(t),$$

where p denotes the longitudinal winding and q the meridional winding.

The second useful property concerns the behavior of these invariants under basic topological operations. Let $CFK^\infty(K)^*$ denote the dual of $CFK^\infty(K)$; we give the precise definition of the dual complex in Section 2. Ozsváth and Szabó [OS04] show that

$$CFK^\infty(K_1 \# K_2) \simeq CFK^\infty(K_1) \otimes CFK^\infty(K_2)$$

and that

$$CFK^\infty(-K) \simeq CFK^\infty(K)^*,$$

allowing us to compute CFK^∞ for linear combinations of L -space knots and their inverses.

With these techniques, we are able to realize a large class of $\mathbb{Z} \oplus \mathbb{Z}$ -filtered chain complexes (up to ε -equivalence), and by studying the structure of the group \mathcal{F} , we can understand where in the filtration these knots lie.

The order type in Theorem 1 is almost certainly not a complete description of the indexing set of \mathcal{F} . One reason for this is that we limited ourselves to linear combinations of a small class of L -space knots for computational reasons. Moreover, to achieve our result, we needed only to consider connected sums of at most two L -space knots, and the L -space knots in question were always cables of torus knots. Sample calculations suggest that with linear combinations of iterated torus knots, a richer order type is possible. An interesting question to consider is whether linear combinations of non- L -space knots would further enlarge the order type.

The results of [Hom11] defined various numerical concordance invariants associated to $CFK^\infty(K)$ that, in a sense, are a refinement of the Ozsváth-Szabó τ invariant [OS03]. This paper studies such

invariants in more depth, giving a better understanding of the relationship between these invariants and the structure of the concordance group.

Organization. We begin in Section 2 with the necessary background on knot Floer homology, totally ordered groups, and L -space knots, including definitions of the invariant ε and the group \mathcal{F} . We proceed to prove algebraic results about \mathcal{F} (Sections 3 and 4) and the existence of a certain family of elements in \mathcal{F} (Section 5) through direct computation. In Section 6, we find knots that allow us to apply our preceding lemmas to understand the order type of \mathcal{F} , which leads to the proof of Theorem 1. We work with coefficients in $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ throughout.

Acknowledgements. The ideas for this paper began during the Summer 2011 Topology REU at Columbia University. The authors would like to thank the other participants in the group, Vivian Josie Bailey and Chun Ye, for their interest in the project, and the organizers of the program for providing the opportunity to work together.

2. BACKGROUND

2.1. The knot Floer complex and concordance. We begin with the necessary background on knot Floer homology, as defined in [OS04] and [Ras03]. To a knot K , we associate a $\mathbb{Z} \oplus \mathbb{Z}$ -filtered, \mathbb{Z} -graded chain complex over the ring $\mathbb{F}[U, U^{-1}]$, where U is a formal variable. The \mathbb{Z} -grading is called the Maslov, or homological, grading. We denote this complex by $CFK^\infty(K)$, and the filtered chain homotopy type of $CFK^\infty(K)$ is an invariant of the knot K . The ordering on $\mathbb{Z} \oplus \mathbb{Z}$ is given by $(i, j) \leq (i', j')$ if $i \leq i'$ and $j \leq j'$.

The differential, ∂ , decreases the homological grading by one and respects the $\mathbb{Z} \oplus \mathbb{Z}$ -filtration. Multiplication by U shifts the \mathbb{Z} -grading by two and decreases the $\mathbb{Z} \oplus \mathbb{Z}$ -filtration by $(1, 1)$. Connected sum of knots corresponds to tensor product of their respective chain complexes, i.e.,

$$CFK^\infty(K_1 \# K_2) \simeq CFK^\infty(K_1) \otimes_{\mathbb{F}[U, U^{-1}]} CFK^\infty(K_2).$$

Taking the reverse of the mirror image of a knot corresponds to taking the dual of the knot Floer complex, i.e.,

$$CFK^\infty(-K) \simeq CFK^\infty(K)^*,$$

where $CFK^\infty(K)^*$ denotes the dual of $CFK^\infty(K)$, that is, $\text{Hom}_{\mathbb{F}[U, U^{-1}]}(CFK^\infty(K), \mathbb{F}[U, U^{-1}])$. The complex $CFK^\infty(K)$ is filtered chain homotopic to the complex obtained by interchanging i and j .

To better understand $\mathbb{Z} \oplus \mathbb{Z}$ -filtered chain complexes, it is convenient to depict them in the (i, j) -plane, where the (i, j) -coordinates depict the filtration level. The \mathbb{Z} -grading is suppressed from this picture. We consider the generators over $\mathbb{F}[U, U^{-1}]$. We place a generator x at the lattice point (i, j) , where (i, j) is the filtration level of x . We use arrows to describe the differential; if y appears with non-zero coefficient in ∂x , then we draw an arrow from x to y . Since the differential respects the $\mathbb{Z} \oplus \mathbb{Z}$ -filtration, the arrows will necessarily point (non-strictly) to the left and down. Up to filtered chain homotopy, one may assume that the differential will strictly decrease the filtration [Ras03, Lemma 4.5], and indeed, that will be the case for all of the complexes we consider. At times, it will be convenient to consider only the part of ∂ that preserves the j - or i -filtration level. We use ∂^{horz} and ∂^{vert} , respectively, to denote these.

The subquotient of $CFK^\infty(K)$ consisting of the $i = 0$ column yields the complex $\widehat{CF}(S^3)$, and so the homology of the $i = 0$ column (or in fact, any column, up to a grading shift) is isomorphic to \mathbb{F} . Similarly, the homology of any row is also isomorphic to \mathbb{F} .

The picture for $CFK^\infty(K)^*$ is closely related to the picture for $CFK^\infty(K)$; one simply reverses the direction of each arrow, as well as both filtrations. (In practice, this may be accomplished by turning the page upside down and reversing the directions of all of the arrows.)

A basis $\{x_k\}$ over $\mathbb{F}[U, U^{-1}]$ for a filtered chain complex C is a *filtered basis* if $\{U^n \cdot x_k \mid U^n \cdot x_k \in C_{i,j}, n \in \mathbb{Z}\}$ is a basis over \mathbb{F} for the subcomplex $C_{i,j}$ for all $i, j \in \mathbb{Z}$, where $C_{i,j}$ denotes the $(i, j)^{\text{th}}$ -filtered subcomplex. In this paper, we will often perform a filtered change of basis, producing a new filtered basis from an old one. Given a filtered basis $\{x_k\}$, we can produce a new filtered basis $\{x'_k\}$, where

$$x'_k = \begin{cases} x_k + x_\ell & k = n \\ x_k & \text{otherwise} \end{cases}$$

for some n and ℓ such that the filtration level of x_ℓ is less than or equal to that of x_n . In other words, one may replace a basis element with itself plus elements of lesser or equal filtration level. We will often omit the prime from the new basis and denote this change of basis by

$$x_n \rightarrow x_n + x_\ell, \quad x_k \rightarrow x_k, \quad k \neq n.$$

A basis $\{x_i\}$ over $\mathbb{F}[U, U^{-1}]$ for $CFK^\infty(K)$ is called *vertically simplified* if for each basis element x_i , exactly one of the following holds:

- There is a unique incoming vertical arrow into x_i .
- There is a unique outgoing vertical arrow from x_i .
- There are no vertical arrows entering or leaving x_i .

Note that since the homology of a column is \mathbb{F} , there is a unique basis element of a vertically simplified basis with no incoming or outgoing vertical arrows, called the *vertically distinguished element*. The analogous definition can be made for a *horizontally simplified* basis. By [LOT08, Proposition 11.52], one may always choose a basis which is vertically simplified, or if one prefers, horizontally simplified.

Given a vertically simplified basis, consider the subquotient complex associated to the $i = 0$ column. The j -coordinate of the vertically distinguished element in this column is a concordance invariant, defined by Ozsváth and Szabó in [OS03] and denoted $\tau(K)$.

While it remains unknown whether a simultaneously vertically and horizontally simplified basis always exists in general, we are able to find such a basis for the complexes under consideration in this paper. Moreover, one may always find a horizontally simplified basis where one of the basis elements, say x_0 , is the distinguished element of some *vertically* simplified basis [Hom12, Lemmas 3.2 and 3.3]. The $\{-1, 0, 1\}$ -valued concordance invariant ε can be defined in terms of such a basis.

Definition 2.1. *The invariant $\varepsilon(K)$ is defined in terms of the above basis for $CFK^\infty(K)$ as follows:*

- (1) $\varepsilon(K) = 1$ if there is a unique incoming horizontal arrow into x_0 .
- (2) $\varepsilon(K) = -1$ if there is a unique outgoing horizontal arrow from x_0 .
- (3) $\varepsilon(K) = 0$ if there are no horizontal arrows entering or leaving x_0 .

To emphasize that ε is in fact an invariant of a bifiltered chain complex, we may at times write $\varepsilon(CFK^\infty(K))$, rather than $\varepsilon(K)$. Alternatively, the invariant ε can be defined in terms of the (non-)vanishing of certain cobordism maps on \widehat{HF} , as in [Hom11, Definition 3.1].

Proposition 2.2 ([Hom12, Proposition 3.6]). *The following are properties of $\varepsilon(K)$:*

- (1) *If K is smoothly slice, then $\varepsilon(K) = 0$.*
- (2) *$\varepsilon(-K) = -\varepsilon(K)$.*

- (3) (a) If $\varepsilon(K) = \varepsilon(K')$, then $\varepsilon(K \# K') = \varepsilon(K) = \varepsilon(K')$.
 (b) If $\varepsilon(K) = 0$, then $\varepsilon(K \# K') = \varepsilon(K')$.

Notice that if K_1 and K_2 are concordant, then $\varepsilon(CFK^\infty(K_1) \otimes CFK^\infty(K_2)^*) = 0$, motivating the following definition.

Definition 2.3. Two bifiltered chain complexes C_1 and C_2 are ε -equivalent, denoted \sim_ε , if

$$\varepsilon(C_1 \otimes C_2^*) = 0.$$

Recall that the monoid of knots under connected sum could be made into a group by quotienting by a suitable equivalence relation. In a similar manner, chain complexes under tensor product form a monoid, and using the idea of ε -equivalence, we can obtain a group.

Definition 2.4. The group \mathcal{F} is

$$\mathcal{F} = (\{CFK^\infty(K) \mid K \subset S^3\}, \otimes) / \sim_\varepsilon.$$

It is clear from the definition of \mathcal{F} and properties of CFK^∞ that we obtain a group homomorphism

$$\mathcal{C} \rightarrow \mathcal{F}$$

by sending $[K]$ to $[CFK^\infty(K)]$. We call this map ϕ . Notice that $\mathcal{F} \cong \mathcal{C} / \ker \phi = \mathcal{C} / \{[K] \mid \varepsilon(K) = 0\}$. For ease of notation, we write

$$\llbracket K \rrbracket$$

to denote $[CFK^\infty(K)]$ and $+$ to denote the group operation. Note that $-\llbracket K \rrbracket = \llbracket -K \rrbracket$. We denote the identity of \mathcal{F} , $\llbracket \text{unknot} \rrbracket$, by 0.

One of the advantages of this approach is that the group \mathcal{F} has a rich algebraic structure. In particular, \mathcal{F} is totally ordered, with the ordering given by

$$[CFK^\infty(K_1)] > [CFK^\infty(K_2)] \iff \varepsilon(CFK^\infty(K_1) \otimes CFK^\infty(K_2)^*) = 1.$$

By considering the behavior of ε under connected sum, it follows that this total ordering is well-defined.

2.2. Totally ordered groups. Two totally ordered sets S_1 and S_2 are *order isomorphic* if there exists a bijection $S_1 \rightarrow S_2$ such that the bijection and its inverse are both order-preserving. The order equivalence class of S is called the *order type* of S .

Given a totally ordered abelian group G , one can naturally define a notion of absolute value, i.e., for any $g \in G$,

$$|g| = \begin{cases} g & \text{if } g \geq \text{id}_G \\ -g & \text{otherwise.} \end{cases}$$

Two elements g and h of a totally ordered abelian group G are said to be *Archimedean equivalent*, denoted \sim_{Ar} , if there exist $m, n \in \mathbb{N}$ such that

$$|g| > m \cdot |h| \quad \text{and} \quad |h| > n \cdot |g|.$$

The set of Archimedean equivalence classes of G inherits an ordering from the group, and the order type of this set is called the *coarse order type* of the group.

Let $[g]_{\text{Ar}}$ denote the Archimedean equivalence class of g . If $[h]_{\text{Ar}} < [g]_{\text{Ar}}$, then $n \cdot |h| < |g|$ for all $n \in \mathbb{N}$, and we write

$$|h| \ll |g|.$$

(If one restricts oneself to only positive elements in the group, then the absolute value signs may be omitted.) For positive $\llbracket K \rrbracket, \llbracket J \rrbracket \in \mathcal{F}$, note that $\llbracket K \rrbracket \gg \llbracket J \rrbracket \iff \varepsilon(K \# -nJ) = 1$ for all $n \in \mathbb{N}$.

A totally ordered group inherits a natural filtration, with the indexing set given by the coarse order type of the group. Given an Archimedean equivalence class, choose a representative g , and consider the subgroup

$$H_g = \{h \in G \mid [h]_{\text{Ar}} \leq [g]_{\text{Ar}}\}.$$

Indeed, it follows from the definition of Archimedean equivalence that the set H_g is closed under the group operation, and it is clear that if $h \in H_g$, then the inverse of h is as well. The filtration now also follows from the definition of Archimedean equivalence, since $[g_1]_{\text{Ar}} < [g_2]_{\text{Ar}}$ implies that $H_{g_1} \subset H_{g_2}$ and that $\mathbb{Z} \subset H_{g_2}/H_{g_1}$, generated by g_2 .

Applying these tools to the group \mathcal{F} , we obtain a filtration on \mathcal{F} , which we may pull back to give a filtration on \mathcal{C} . The effectiveness of this approach is largely determined by the coarse order type of \mathcal{F} . For instance, it is easily seen [Hom11, Proposition 4.8] that the coarse order type of \mathcal{F} contains ω as an ordered subset, with $\mathcal{F}_n := \phi^{-1}[H_{\llbracket T_{n,n+1} \rrbracket}]$ giving a filtration on \mathcal{C} indexed by \mathbb{N} . This is precisely the reversed order type of the n -solvable and bipolar filtrations. Our goal is to achieve an indexing set with finer order type.

2.3. L -space knots. One of our main tools for computing the knot Floer complex of large families of knots concerns L -space knots. Recall that an L -space Y is a rational homology sphere for which $\text{rk } \widehat{HF}(Y) = |H_1(Y; \mathbb{Z})|$, and that an L -space knot is a knot on which some positive surgery is an L -space. In [OS05, Theorem 1.2], Ozsváth and Szabó show that if a knot K admits a positive L -space surgery, then its knot Floer complex is completely determined by the Alexander polynomial of K . In particular, if K is an L -space knot, then the Alexander polynomial of K is of the form

$$\Delta_K(t) = \sum_{i=0}^M (-1)^i t^{n_i}$$

for an even positive integer M and some increasing sequence of $n_i \in \mathbb{Z}_{\geq 0}$ satisfying the symmetry requirement that

$$n_i + n_{M-i} = 2g(K),$$

where $g(K)$ is the genus of K and we have normalized the Alexander polynomial to have a constant term and no negative exponents, i.e., $n_0 = 0$.

The sequence n_i determines the knot Floer complex of K . A filtered basis over $\mathbb{F}[U, U^{-1}]$ for $CFK^\infty(K)$ is given by $\{x_i\}$, $i = 0, \dots, M$, with the following differentials:

$$\partial x_i = \begin{cases} x_{i-1} + x_{i+1} & i \text{ odd} \\ 0 & i \text{ even,} \end{cases}$$

where the arrow from x_i to x_{i-1} is horizontal of length $n_i - n_{i-1}$, and the arrow from x_i to x_{i+1} is vertical of length $n_{i+1} - n_i$. See Figure 1 for an example.

In the next section, we introduce special notation for denoting certain families of bifiltered chain complexes, with the above result taking the form

$$\llbracket K \rrbracket = \llbracket (n_i - n_{i-1})_{i=1}^{M/2} \rrbracket$$

in that notation. Notice that in the (i, j) -plane, this complex has the appearance of a “staircase”. (In fact, having such a staircase complex is a necessary and sufficient condition for being an L -space knot [OS05].) Such complexes will play a key role in this paper.

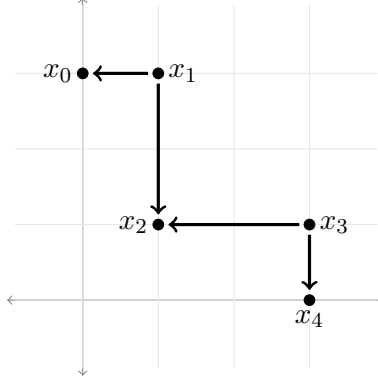


FIGURE 1. A basis for $CFK^\infty(T_{3,4})$, where $T_{3,4}$ denotes the $(3,4)$ -torus knot. The Alexander polynomial of $T_{3,4}$ is $\Delta_{T_{3,4}}(t) = 1 - t + t^3 - t^5 + t^6$.

2.4. Notation for chain complex classes. Consider the monoid $(M, \otimes_{\mathbb{F}[U, U^{-1}]})$ of all bifiltered chain complexes with “vertical” homology \mathbb{F} that are chain homotopic to the bifiltered chain complex obtained by interchanging i and j .

Definition 2.5. The group \mathcal{F}_{alg} is defined to be

$$\mathcal{F}_{\text{alg}} = (M, \otimes) / \sim_\varepsilon.$$

Clearly $\mathcal{F} \subseteq \mathcal{F}_{\text{alg}}$ since $\{CFK^\infty(K)\} \subseteq M$. It is known that $\{CFK^\infty(K)\} \neq M$, since there does not exist an L -space knot K with $a_1(K) > 1$ [Ras04, Theorem 2.3]; see below for the definition of a_1 . However, it is an open question whether $\mathcal{F} = \mathcal{F}_{\text{alg}}$.

For the remainder of the section, we assume that our complexes admit a simultaneously vertically and horizontally simplified basis. Given such a basis B for C , $[C] \in \mathcal{F}_{\text{alg}}$, we can uniquely trace C from the vertically distinguished element to the horizontally distinguished element along horizontal and vertical arrows. Denote the i^{th} basis element encountered along this path by x_{i-1} . Then if x_{2m} is horizontally distinguished, the complex C_{red} consisting of the basis elements $\{x_i\}$, $i = 0, \dots, 2m$, together with the arrows between them, is ε -equivalent to C . We will focus on the case when there are no arrows between $\{x_i\}$ and $B \setminus \{x_i\}$ so that $C = C_{\text{red}} \oplus C_A$, where C_A is an acyclic complex, i.e., its total homology, $H_*(C_A)$, is zero. (C_A must be acyclic since the total homology of C and C_{red} both have rank one.) It is clear from the definition of ε in [Hom11, Definition 3.1] that

$$C_{\text{red}} \oplus C_A \sim_\varepsilon C_{\text{red}}.$$

For i odd, let $|a_i|$ be the length of the horizontal arrow between x_i and x_{i-1} , with a_i positive if the arrow is outgoing from x_i , otherwise negative. For i even, let $|a_i|$ be the length of the vertical arrow between x_i and x_{i-1} , with a_i positive if the arrow is incoming to x_i , otherwise negative. If $m > 0$, we write

$$[a_1, a_2, \dots, a_m]$$

to denote $[C]$ and $+$ to denote the group operation. Note that $-[C] = [-a_1, -a_2, \dots, -a_m]$. We denote the identity of \mathcal{F}_{alg} , $[\]$, by 0. It may be shown (cf. [Hom11, Lemma 6.1]) that if defined, a_i is an invariant of the class $[C]$. Hence each representative of $[C]$ must have at least $2m + 1$ basis elements. Moreover, only one representative has basis size exactly $2m + 1$, and we call such a complex *reduced*. See Figure 2 for two examples.

Given an integer sequence $(a_i)_{i=1}^m$, $[(a_i)_{i=1}^m] \in \mathcal{F}_{\text{alg}}$, it follows that $a_i = a_{2m+1-i}$ by the diagonal symmetry of each complex in M , and so a_i is implicitly defined for $i = m + 1, \dots, 2m$ as well. If

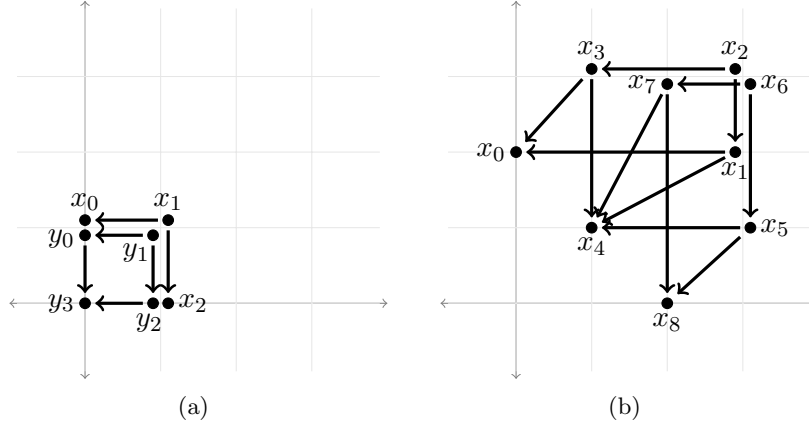


FIGURE 2. Left, a basis for $CFK^\infty(5_2)$, showing $[CFK^\infty(5_2)] = [1]$. Right, a basis for the reduced representative of $[3, -1, -2, 2]$.

some $a_i = 0$, it may be removed with the convention $[\dots, a_{i-1}, 0, a_{i+1}, \dots] = [\dots, a_{i-1} + a_{i+1}, \dots]$. We assume in the remainder of the section that any zero entries have been removed. We write $C \in [(a_i)_{i=1}^m]$ to denote that C is a representative of the the class $[(a_i)_{i=1}^m]$. If $(a_i)_{i=1}^m$ has mixed positive and negative entries, the necessary set of diagonal arrows that ensure $\partial^2 = 0$ are assumed to be present in each $C \in [(a_i)_{i=1}^m]$. Note this is not always possible for an arbitrary finite integer sequence $(a_i)_{i=1}^m$, so not every $[(a_i)_{i=1}^m]$ is in \mathcal{F}_{alg} . For example, $[1, -2]$ does not admit a chain complex representative; there is no collection of diagonal arrows that makes $\partial^2 = 0$.

If C_1 is a representative of the class $[a_1, a_2, \dots, a_m]$ with basis $\{x_i\}$, $i = 0, \dots, 2m$, then C_1 must be the reduced representative, and we assume the x_i are labeled in the order described above. Furthermore, we assume C_1^* has basis $\{x_i^*\}$ labeled in this order; that is, x_0^* is the vertically distinguished element. Finally, given $C_2 \in [b_1, b_2, \dots, b_n]$ with basis $\{y_j\}$, $j = 0, \dots, 2n$, then $(C_1 \otimes C_2)$ naturally has (unsimplified) basis $\{x_i y_j\}$, where $x_i y_j = x_i \otimes y_j$.

It is easily seen that

$$\partial^{\text{horz}} x_i = \begin{cases} x_{i-1} & i \text{ odd, } a_i > 0 \\ x_{i+1} & i \text{ even, } i \neq 2m, a_{i+1} < 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \partial^{\text{vert}} x_i = \begin{cases} x_{i+1} & i \text{ odd, } a_{i+1} > 0 \\ x_{i-1} & i \text{ even, } i \neq 0, a_i < 0 \\ 0 & \text{otherwise.} \end{cases}$$

If each $a_i > 0$, then C_1 is a staircase complex, i.e., no diagonal arrows are present, and we have

$$(2.1) \quad \partial x_i = \begin{cases} x_{i-1} + x_{i+1} & i \text{ odd} \\ 0 & i \text{ even} \end{cases} \quad \text{and} \quad \partial x_i^* = \begin{cases} 0 & i \text{ odd} \\ x_{i-1}^* + x_{i+1}^* & i \text{ even, } i \neq m \mp m \\ x_{i \pm 1}^* & i = m \mp m. \end{cases}$$

Using $fl(x_i)$ to denote the filtration level of x_i , we have $fl(x_0) = (0, \tau(C_1))$. It is further clear that

$$(2.2) \quad fl(x_i) = \begin{cases} fl(x_{i-1}) + (a_i, 0) & i \text{ odd} \\ fl(x_{i-1}) - (0, a_i) & i \text{ even} \end{cases} \quad \text{and} \quad fl(x_i^*) = -fl(x_i).$$

By definition of the operation of tensor product on chain complexes, we also have

$$(2.3) \quad \partial(x_i y_j) = \partial(x_i) y_j + x_i \partial(y_j)$$

$$(2.4) \quad fl(x_i y_j) = fl(x_i) + fl(y_j).$$

Notice $fl(x_0^*) = -(0, \tau(C_1)) = (0, \tau(C_1^*))$ and $fl(x_0 y_0) = (0, \tau(C_1)) + (0, \tau(C_2)) = (0, \tau(C_1 \otimes C_2))$.

Finally, note that the concordance invariants $\tau(K)$ of [OS03] and $\varepsilon(K)$, $a_1(K)$, and $a_2(K)$ of [Hom11] are well-defined invariants of elements in \mathcal{F}_{alg} . For $[C] = [a_1, a_2, \dots, a_m] \in \mathcal{F}_{\text{alg}}$, we have

$$\tau(C) = \sum_{i=1}^m a_i \quad \text{and} \quad \varepsilon(C) = \begin{cases} \text{sgn}(a_1) & \text{if } m > 0 \\ 0 & \text{otherwise,} \end{cases}$$

and if $\varepsilon(C) = 1$,

$$a_1(C) = a_1 \quad \text{and} \quad a_2(C) = \begin{cases} a_2 & \text{if } a_2 > 0 \\ \text{undefined} & \text{otherwise.} \end{cases}$$

3. TENSOR PRODUCTS OF STAIRCASE COMPLEXES

For the calculations that follow in Sections 3 and 4, we suppress the U -translates from this picture, which is always possible for the complexes under consideration here, given an appropriate choice of basis. With this convention, tensor products will now take place over the ground field \mathbb{F} , rather than the ring $\mathbb{F}[U, U^{-1}]$.

In this section, we prove two lemmas on the group operation of \mathcal{F}_{alg} . Our approach is to take the tensor product of two complexes, both reduced with $\varepsilon = 1$, then vertically and horizontally simplify the basis of the product to determine the reduced representative of its ε -equivalence class.

Lemma 3.1. *Let $a_i, b_j > 0$. If m is even and $\max\{a_i | i \text{ odd}\} \leq b_j \leq \min\{a_i | i \text{ even}\}$ for $i = 1, \dots, m$ and $j = 1, \dots, n$, then*

$$[a_1, a_2, \dots, a_m] + [b_1, b_2, \dots, b_n] = [a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_n].$$

Proof. For $C_1 \in [(a_i)_{i=1}^m]$ with basis $\{x_i\}$, $i = 0, \dots, 2m$, $C_2 \in [(b_j)_{j=1}^n]$ with basis $\{y_j\}$, $j = 0, \dots, 2n$, and $P \in [(a_i)_{i=1}^m, (b_j)_{j=1}^n]$ reduced, we prove $C_1 \otimes C_2 \sim_\varepsilon P$. Define the sets

$$S = \{x_i y_0 \mid i < m\} \cup \{x_m y_j \mid i > m\}$$

$$B_{i,j} = \begin{cases} \{x_i y_j, x_{i-1} y_j, x_i y_{j+1}, x_{i-1} y_{j+1}\} & i, j \text{ odd, } i < m \\ \{x_i y_j, x_{i+1} y_j, x_i y_{j-1}, x_{i+1} y_{j-1}\} & i, j \text{ odd, } i > m \end{cases}$$

Supposing m is even, $\cup_{i < m, j} B_{i,j} = \{x_i y_j \mid i < m, j \neq 0\}$ and $\cup_{i > m, j} B_{i,j} = \{x_i y_j \mid i > m, j \neq 2n\}$. Therefore, $\{x_i y_j\} = S \cup \{\cup_{i,j} B_{i,j}\}$. Note that S is the elements $\{x_i y_0 \mid i < m\} \cup \{x_m y_j \mid j \leq n\}$ along with their reflection about the diagonal. Given

$$(3.1) \quad \begin{aligned} \partial(x_i y_0) &= \begin{cases} x_{i-1} y_0 + x_{i+1} y_0 & i \text{ odd} \\ 0 & i \text{ even} \end{cases} & fl(x_i y_0) &= \begin{cases} fl(x_{i-1} y_0) + (a_i, 0) & i \text{ odd} \\ fl(x_{i-1} y_0) - (0, a_i) & i \text{ even} \end{cases} \\ \partial(x_m y_j) &= \begin{cases} x_m y_{j-1} + x_m y_{j+1} & j \text{ odd} \\ 0 & j \text{ even} \end{cases} & fl(x_m y_j) &= \begin{cases} fl(x_m y_{j-1}) + (b_j, 0) & j \text{ odd} \\ fl(x_m y_{j-1}) - (0, b_j) & j \text{ even,} \end{cases} \end{aligned}$$

it is clear that S is a basis for P . We show that each of the remaining mn sets of elements $B_{i,j}$ forms an acyclic summand under a filtered change of basis. These summands look like boxes—see Figure 3 for an example. Since $B_{i,j} \rightarrow B_{2m-i, 2n-j}$ when $x_i y_j \rightarrow x_{2m-i} y_{2n-j}$, the proofs for $i < m$ and $i > m$ are the same under the transformation $x_i y_j \rightarrow x_{2m-i} y_{2n-j}$, $a_i \rightarrow a_{2m+1-i}$, and $b_j \rightarrow b_{2n+1-j}$ by the diagonal symmetry of P . It therefore suffices to redefine the basis for $i = 0, \dots, m-1$.

For all i and j odd, produce the new basis

$$(3.2) \quad x_{i-1} y_j \rightarrow x_{i-1} y_j + x_i y_{j-1}, \quad x_i y_{j+1} \rightarrow x_i y_{j+1} + x_{i+1} y_j, \quad x_{i-1} y_{j+1} \rightarrow x_{i-1} y_{j+1} + x_{i+1} y_{j-1}.$$

It follows that $B_{i,j} \rightarrow B'_{i,j}$, where $B'_{i,j} = \{x_i y_j, x_{i-1} y_j + x_i y_{j-1}, x_i y_{j+1} + x_{i+1} y_j, x_{i-1} y_{j+1} + x_{i+1} y_{j-1}\}$.

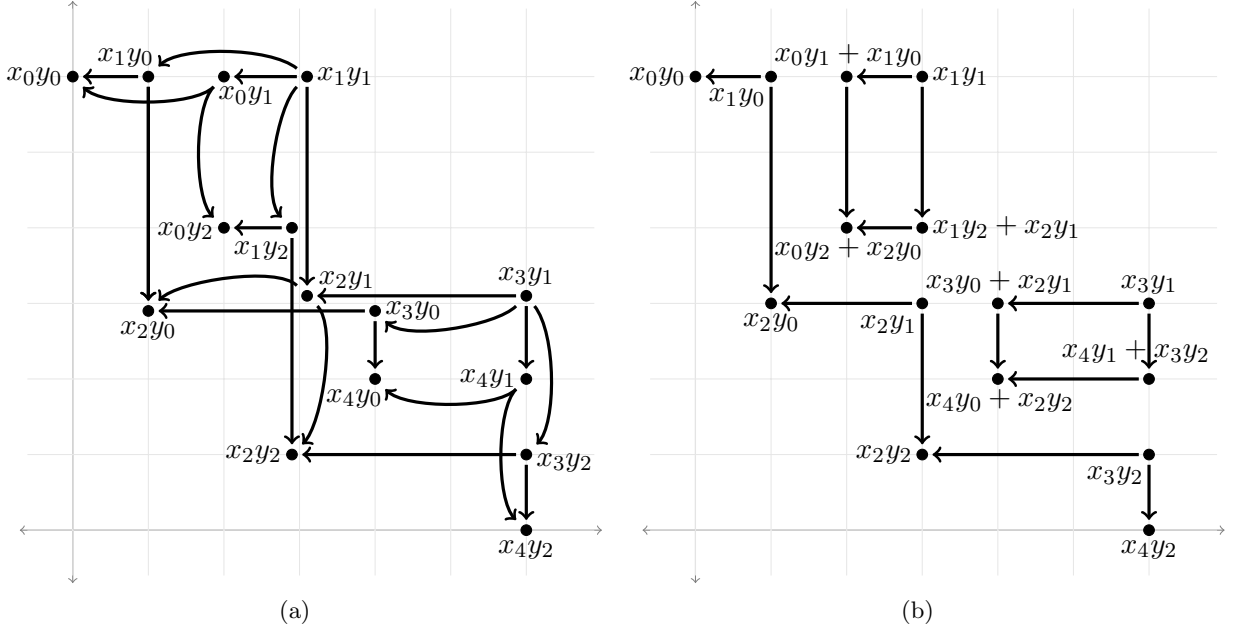


FIGURE 3. $C_1 \otimes C_2$ with, left, basis $\{x_i y_j\}$ and, right, simplified basis $S \cup B'_{1,1} \cup B'_{3,1}$, where $C_1 \in [1, 3]$ has basis $\{x_0, x_1, x_2, x_3, x_4\}$ and $C_2 \in [2]$ has basis $\{y_0, y_1, y_2\}$.

First, we check that the differentials on $B'_{i,j}$ give an acyclic summand. Using (2.1, 2.3) with i and j odd and coefficients in $\mathbb{Z}/2\mathbb{Z}$,

$$\begin{aligned}
 \partial(x_i y_j) &= x_{i-1} y_j + x_{i+1} y_j + x_i y_{j-1} + x_i y_{j+1} \\
 &= (x_{i-1} y_j + x_i y_{j-1}) + (x_i y_{j+1} + x_{i+1} y_j) \\
 \partial(x_{i-1} y_j + x_i y_{j-1}) &= (x_{i-1} y_{j-1} + x_{i-1} y_{j+1}) + (x_{i-1} y_{j-1} + x_{i+1} y_{j-1}) \\
 &= x_{i-1} y_{j+1} + x_{i+1} y_{j-1} \\
 \partial(x_i y_{j+1} + x_{i+1} y_j) &= (x_{i-1} y_{j+1} + x_{i+1} y_{j+1}) + (x_{i+1} y_{j-1} + x_{i+1} y_{j+1}) \\
 &= x_{i-1} y_{j+1} + x_{i+1} y_{j-1} \\
 \partial(x_{i-1} y_{j+1} + x_{i+1} y_{j-1}) &= 0.
 \end{aligned}
 \tag{3.3}$$

That is, both $(x_{i-1} y_j + x_i y_{j-1})$ and $(x_i y_{j+1} + x_{i+1} y_j)$ have one incoming arrow from $x_i y_j$ and one outgoing arrow to $(x_{i-1} y_{j+1} + x_{i+1} y_{j-1})$, joining the former to the latter. As there are no outgoing arrows from S to $\{x_i y_j\} \setminus S$ or $B'_{i,j}$ to $\{x_i y_j\} \setminus B_{i,j}$, neither are there incoming arrows to $B'_{i,j}$. We conclude that each $B'_{i,j}$ is a basis for an acyclic subcomplex that is split off from the rest of P .

Second, we check that our chosen basis respects the filtration. Using (2.2, 2.4) with i and j odd,

$$\begin{aligned}
 fl(x_{i-1} y_j) &= (fl(x_i) - (a_i, 0)) + (fl(y_{j-1}) + (b_j, 0)) \\
 &= fl(x_i y_{j-1}) + (b_j - a_i, 0) \\
 fl(x_i y_{j+1}) &= (fl(x_{i+1}) + (0, a_{i+1})) + (fl(y_j) - (0, b_{j+1})) \\
 &= fl(x_{i+1} y_j) + (0, a_{i+1} - b_{j+1}) \\
 fl(x_{i-1} y_{j+1}) &= (fl(x_{i+1}) + (-a_i, a_{i+1})) + (fl(y_{j-1}) + (b_j, -b_{j+1})) \\
 &= fl(x_{i+1} y_{j-1}) + (b_j - a_i, a_{i+1} - b_{j+1}).
 \end{aligned}
 \tag{3.4}$$

We require $fl(x_{i-1}y_j) \geq fl(x_{i-1}y_{j-1})$, $fl(x_{i-1}y_{j+1}) \geq fl(x_{i+1}y_j)$, and $fl(x_{i-1}y_{j+1}) \geq fl(x_{i+1}y_{j-1})$. By (3.4), this is equivalent to $a_i \leq b_j$ and $b_{j+1} \leq a_{i+1}$ for i and j odd, which in turn is equivalent to $a_i \leq b_j \leq a_{i+1}$ for i odd and $j = 1, \dots, n$, which is true by hypothesis. \square

Lemma 3.2. *Let $a > 0$ and $c_\ell, d, p_\ell, q \geq 0$. If $q \geq \max\{p_\ell\}$ and $d \leq \min\{c_\ell\}$ for $\ell = 1, \dots, r$, then*

$$[((1, a)_1^{p_\ell}, 1, a + c_\ell)_{\ell=1}^r] + [(1, a)_1^q, 1, a + d] = [((1, a)_1^{p_\ell}, 1, a + c_\ell)_{\ell=1}^r, (1, a)_1^q, 1, a + d].$$

Proof. Set $m_\ell = 2 \sum_{k=1}^\ell p_k$ and $n = 2q$. For $C_1 \in [((1, a)_1^{p_\ell}, 0, c_\ell)_{\ell=1}^r]$ with basis $\{x_i\}$, $i = 0, \dots, 2m_r$, $C_2 \in [(1, a)_1^q, 0, d]$ with basis $\{y_j\}$, $j = 0, \dots, 2n$, and $P \in [((1, a)_1^{p_\ell}, 0, c_\ell)_{\ell=1}^r, (1, a)_1^q, 0, d]$ reduced, we prove $C_1 \otimes C_2 \sim_\varepsilon P$. Using $p_k = p_{2r-k+1}$, let $\ell = 0, \dots, 2r$. Define the sets

$$\begin{aligned} S &= \{x_i y_0 \mid i < m_r\} \cup \{x_{m_r} y_j\} \cup \{x_i y_{2n} \mid i > m_r\} \\ B_{i,j} &= \begin{cases} \{x_i y_j, x_{i-1} y_j, x_i y_{j+1}, x_{i-1} y_{j+1}\} & i, j \text{ odd}, i < m_r, j \neq n-1 \\ \{x_i y_j, x_{i+1} y_j, x_i y_{j-1}, x_{i+1} y_{j-1}\} & i, j \text{ odd}, i > m_r, j \neq n+1 \end{cases} \\ R_\ell &= \begin{cases} \{x_i y_{n-1}, x_i y_n \mid m_\ell \leq i < m_{\ell+1}\} & \ell < r \\ \{x_i y_{n+1}, x_i y_n \mid m_\ell \geq i > m_{\ell-1}\} & \ell > r. \end{cases} \end{aligned}$$

Reasoning as in the preceding proof, $\{x_i y_j\} = S \cup \{\cup_{i,j} B_{i,j}\} \cup \{\cup_\ell R_\ell\}$, and S is a basis for P . We show that each of the remaining $m_r(n-2)$ sets of elements $B_{i,j}$ and $2r$ sets of elements R_ℓ forms an acyclic summand under a filtered change of basis. The former summands look like boxes and the latter look like rectilinear polygons with $4p_k$ sides—see Figure 4 for an example. Since $B_{i,j} \rightarrow B_{2m_r-i, 2n-j}$ and $R_\ell \rightarrow R_{2r-\ell}$ when $x_i y_j \rightarrow x_{2m_r-i} y_{2n-j}$, it suffices to redefine the basis for $i = 0, \dots, m_r - 1$ and $\ell = 0, \dots, r - 1$.

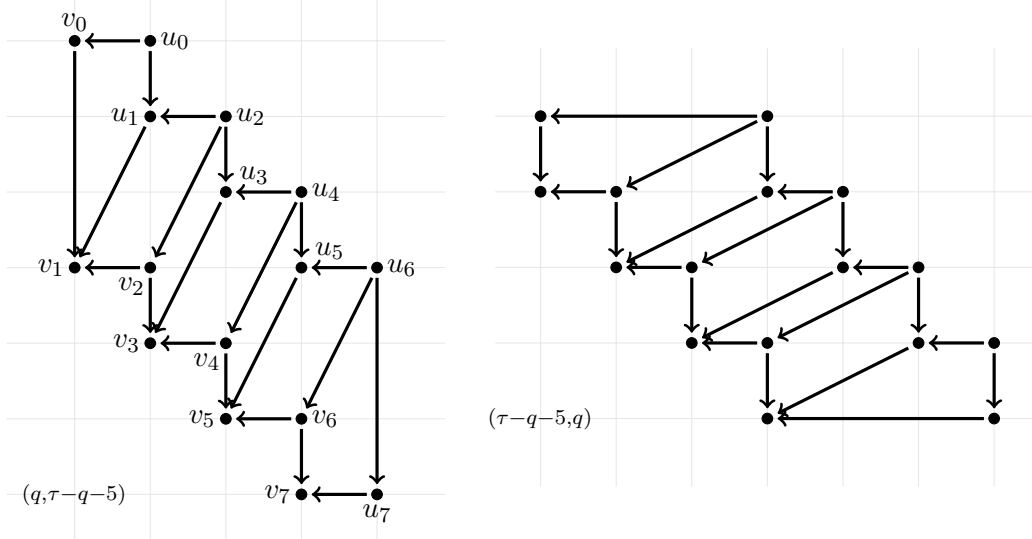


FIGURE 4. Acyclic summands of $C_1 \otimes C_2$ with bases, left, R'_0 and, right, R'_2 , where $C_1 \in [(1, 1)_1^4, 0, c]$ and $C_2 \in [(1, 1)_1^q, 0, 2]$ are reduced with $2 \leq c$ and $q \geq 4$.

Denote the length between x_{i-1} and x_i by a_i and between y_{j-1} and y_j by b_j . For i and j odd, $\max\{a_i\} = 1$ and $\min\{a_{i+1}\} = a$ so that $a_i \leq b_j$ and $b_{j+1} \leq a_{i+1}$ when $j \neq n-1$. Hence by (3.4), we can make the basis change (3.2) for each $B_{i,j}$. We saw that each $B'_{i,j}$ is a basis for a box subcomplex.

Let $m(i) = \min\{m_k | m_k > i\}$. Define the basis element sums

$$(3.5) \quad u_i = \begin{cases} \sum_{k=0}^{m-i} x_{i+k} y_{n-k} & i \text{ odd} \\ \sum_{\substack{k=1 \\ k \text{ odd}}}^{m-i-1} x_{i+k} y_{n-k} & i \text{ even} \end{cases} \quad \text{and} \quad v_{i+1} = \begin{cases} \sum_{k=1}^{m(i+1)-m} x_{i+k} y_{n-k} & i+1 = m \\ x_i y_n + x_m y_{n-m+i} & i+1 \neq m, \end{cases}$$

where $m = m(i)$. Supposing $q \geq p_{\ell+1}$ for all ℓ , we may produce the new basis

$$(3.6) \quad x_i y_{n-1} \rightarrow \begin{cases} u_{i-1} & i \neq m_\ell \\ v_i & i = m_\ell \end{cases} \quad \text{and} \quad x_i y_n \rightarrow \begin{cases} u_i & i+1 = m_{\ell+1} \\ v_{i+1} & i+1 \neq m_{\ell+1} \end{cases}$$

for all i . It follows that $R_\ell \rightarrow R'_\ell$, where $R'_\ell = \{u_i, v_i \mid m_\ell \leq i < m_{\ell+1}\}$.

First, we check that the differentials on R'_ℓ give an acyclic summand. Using (2.1, 2.3) with i odd, ∂u_i for each i and ∂v_{i+1} for each $i+1 \neq m_\ell$ are

$$\begin{aligned} \sum_{k=0}^{m-i} x_{i+k} y_{n-k} &= \partial \sum_{\substack{k=0 \\ k \text{ even}}}^{m-i-1} (x_{i+k} y_{n-k} + x_{i+k+1} y_{n-k-1}) \\ &= \sum_{\substack{k=0 \\ k \text{ even}}}^{m-i-1} ((x_{i+k-1} y_{n-k} + x_{i+k+1} y_{n-k}) + (x_{i+k+1} y_{n-k-2} + x_{i+k+1} y_{n-k})) \\ &= (x_{i-1} y_n + x_m y_{n-m+i-1}) + \sum_{\substack{k=2 \\ k \text{ even}}}^{m-i-1} 2x_{i+k-1} y_{n-k} + \sum_{\substack{k=0 \\ k \text{ even}}}^{m-i-1} 2x_{i+k+1} y_{n-k} \\ \partial(x_i y_n + x_m y_{n-m+i}) &= (x_{i-1} y_n + x_{i+1} y_n) + (x_m y_{n-m+i-1} + x_m y_{n-m+i+1}) \\ &= (x_{i-1} y_n + x_m y_{n-m+i-1}) + (x_{i+1} y_n + x_m y_{n-m+i+1}), \end{aligned}$$

while $\partial v_{m_\ell} = v_{m_\ell+1}$. Using (2.1, 2.3) with i even, ∂u_i for each $i \neq m_\ell$ and ∂v_{i+1} for each i are

$$\begin{aligned} \sum_{\substack{k=1 \\ k \text{ odd}}}^{m-i-1} x_{i+k} y_{n-k} &= \sum_{\substack{k=1 \\ k \text{ odd}}}^{m-i-1} (x_{i+k-1} y_{n-k} + x_{i+k+1} y_{n-k} + x_{i+k} y_{n-k-1} + x_{i+k} y_{n-k+1}) \\ &= \sum_{k=1}^{m-i} x_{i+k-1} y_{n-k} + \sum_{k=0}^{m-i-1} x_{i+k+1} y_{n-k} \\ &= -(x_{i-1} y_n + x_m y_{n-m+i-1}) + \sum_{k=0}^{m-(i-1)} x_{i-1+k} y_{n-k} + \sum_{k=0}^{m-(i+1)} x_{i+1+k} y_{n-k} \\ \partial(x_i y_n + x_m y_{n-m+i}) &= 0, \end{aligned}$$

while $\partial u_{m_\ell} = v_{m_\ell} + u_{m_\ell+1}$. Applying our definitions (3.5) and $\mathbb{Z}/2\mathbb{Z}$ coefficients, we have shown

$$(3.7) \quad \partial u_i = \begin{cases} v_i + u_{i+1} & i = m_\ell \\ v_i + u_{i-1} + u_{i+1} & i \text{ even}, i \neq m_\ell \\ v_i & i \text{ odd} \end{cases} \quad \text{and} \quad \partial v_i = \begin{cases} v_{i+1} & i = m_\ell \\ v_{i-1} + v_{i+1} & i \text{ even}, i \neq m_\ell \\ 0 & i \text{ odd}. \end{cases}$$

That is, R'_ℓ is a basis for two staircases, joined at their endpoints $\{u_{m_\ell}, u_{m_{\ell+1}-1}\}$ and $\{v_{m_\ell}, v_{m_{\ell+1}-1}\}$ by an arrow from u_i to v_i . (Note the diagonal arrows from u_i to v_i for $m_\ell < i < m_{\ell+1} - 1$ ensure $\partial^2 = 0$.) As there are no outgoing arrows from S to $\{x_i y_j\} \setminus S$, $B'_{i,j}$ to $\{x_i y_j\} \setminus B_{i,j}$, or R'_ℓ to $\{x_i y_j\} \setminus R_\ell$, neither are there incoming arrows to $B'_{i,j}$ or R'_ℓ . We conclude that each $B'_{i,j}$ and R'_ℓ is a basis for an acyclic subcomplex that is split off from the rest of P .

Second, we check that our basis respects the filtration. Let $1 < k < m - i$, where $m = m(i + 1)$. For $m = m_{\ell+1}$, (2.2, 2.4) with i odd yields

$$\begin{aligned} fl(x_{i+1}y_{n-1}) &= (fl(x_i) - (0, a)) + (fl(y_n) + (0, a + d)) \quad i + 1 \neq m_\ell \\ &= fl(x_i y_n) + (0, d) \quad i + 1 \neq m_\ell \\ fl(x_{i+k}y_{n-k}) &= \begin{cases} (fl(x_{i+k-1}) - (0, a)) + (fl(y_{n-k+1}) + (0, a)) & k \text{ odd} \\ (fl(x_{i+k-1}) + (1, 0)) + (fl(y_{n-k+1}) - (1, 0)) & k \text{ even} \end{cases} \\ &= fl(x_{i+k-1}y_{n-k+1}) \\ fl(x_m y_{n-m+i}) &= (fl(x_{m-1}) - (0, a + c_{\ell+1})) + (fl(y_{n-m+i+1}) + (0, a)) \quad i + 1 \neq m_\ell \\ &= fl(x_{m-1}y_{n-m+i+1}) - (0, c_{\ell+1}) \quad i + 1 \neq m_\ell \end{aligned}$$

and $fl(x_{m-1}y_n) = fl(x_m y_{n-1}) + (0, c_{\ell+1} - d)$. For i even,

$$\begin{aligned} fl(x_{i+1}y_{n-1}) &= (fl(x_i) + (1, 0)) + (fl(y_n) + (0, a + d)) \\ &= fl(x_i y_n) + (1, a + d) \\ fl(x_{i+k}y_{n-k}) &= (fl(x_{i+k-2}) + (1, -a)) + (fl(y_{n-k+2}) + (-1, a)) \quad k \text{ odd} \\ &= fl(x_{i+k-2}y_{n-k+2}) \quad k \text{ odd} \\ fl(x_m y_{n-m+i}) &= (fl(x_{m-1}) - (0, a + c_{\ell+1})) + (fl(y_{n-m+i+1}) - (1, 0)) \\ &= fl(x_{m-1}y_{n-m+i+1}) - (1, a + c_{\ell+1}). \end{aligned}$$

Letting $i \rightarrow i - 1$, it follows that

$$(3.8) \quad \begin{aligned} fl(x_i y_{n-1}) &= \begin{cases} fl(x_{i-1+k}y_{n-k}) & i \text{ even or } i, k \text{ odd} \\ fl(x_{i-1}y_n) + (0, d) = fl(x_m y_{n-m+i-1}) + (0, c_{\ell+1}) & i \text{ even, } i \neq m_\ell \end{cases} \\ fl(x_i y_n) &= fl(x_m y_{n-m+i}) + (0, c_{\ell+1} - d). \end{aligned}$$

We require $fl(x_i y_n) \geq fl(x_m y_{n-m+i})$ and $fl(x_i y_{n-1}) \geq fl(x_{i-1+k}y_{n-k})$ for each k summed over in u_{i-1} ($i \neq m_\ell$) and v_{m_ℓ} . By (3.8), this is equivalent to $0 \leq d \leq c_{\ell+1}$, which is true by hypothesis. \square

4. ORDERING OF STAIRCASE COMPLEX CLASSES

We now study the ordering on \mathcal{F}_{alg} . We take the tensor product of a complex with n times the dual of another, both reduced with $\varepsilon = 1$, then partially vertically and horizontally simplify the basis of the product to determine its ε value.

Lemma 4.1 ([Hom11, Lemmas 6.3 and 6.4]). *Let $a_i, b_j > 0$. If $b_1 > a_1$ or if $b_1 = a_1$ and $b_2 < a_2$, then*

$$[a_1, a_2, \dots, a_m] \gg [b_1, b_2, \dots, b_n].$$

Lemma 4.2. *Let $a, c > 0$ and $d, p, q \geq 0$. If $q > p$ or if $q = p$ and $d < c$, then*

$$[(1, a)_1^p, 1, a + c] \gg [(1, a)_1^q, 1, a + d].$$

Proof. Set $m = 2p$ and $n = 2q$. For $C \in [(1, a)_1^p, 0, c]$ with basis $\{x_i\}$, $i = 0, \dots, 2m$, and $D_r \in [(1, a)_1^q, 0, d]_1^r$ with basis $\{y_j\}$, $j = 0, \dots, 2nr$, we prove $\varepsilon(C \otimes rD_1^*) = 1$ for all $r \in \mathbb{N}$. By inductive use of Lemma 3.2, $rD_1 \sim_\varepsilon D_r$, and so we may work with the simpler complexes D_r , i.e., we show $\varepsilon(C \otimes D_r^*) = 1$.

By [Hom11, Section 3], Definition 2.1 (1) is equivalent to finding a basis with element u_0 that is the distinguished element of some vertically simplified basis for $(C \otimes D_r^*)$ and in the image of the

horizontal differential. Define the basis element sums

$$(4.1) \quad u_i = \begin{cases} \sum_{k=0}^{m-i} x_{i+k} y_k^* & i \text{ even} \\ \sum_{\substack{k=0 \\ k \text{ even}}}^{m-i-1} x_{i+k} y_k^* & i \text{ odd.} \end{cases}$$

Supposing $q \geq p$, we may produce the new basis $x_i y_0^* \rightarrow u_i$ for $i = 0, \dots, m$. Clearly $u_0 = (x_0 y_0^*)'$ has no incoming vertical arrows since $x_0 y_0^*$ has none, so it suffices to show $\partial^{\text{vert}} u_0 = 0$ and $\partial^{\text{horz}} u_1 = u_0$. That is, we need only partially simplify the basis.

First, we check differentials. By (2.1, 2.3) with i even,

$$\begin{aligned} \partial \sum_{k=0}^{m-i} x_{i+k} y_k^* &= \partial(x_i y_0^*) + \partial \sum_{\substack{k=1 \\ k \text{ odd}}}^{m-i-1} (x_{i+k} y_k^* + x_{i+k+1} y_{k+1}^*) \\ &= x_i y_1^* + \sum_{\substack{k=1 \\ k \text{ odd}}}^{m-i-1} ((x_{i+k-1} y_k^* + x_{i+k+1} y_k^*) + (x_{i+k+1} y_{k+2}^* + x_{i+k+1} y_k^*)) \\ &= x_m y_{m-i+1}^* + \sum_{\substack{k=1 \\ k \text{ odd}}}^{m-i-1} 2x_{i+k-1} y_k^* + \sum_{\substack{k=1 \\ k \text{ odd}}}^{m-i-1} 2x_{i+k+1} y_k^*. \end{aligned}$$

For i odd,

$$\begin{aligned} \partial \sum_{\substack{k=0 \\ k \text{ even}}}^{m-i-1} x_{i+k} y_k^* &= \partial(x_i y_0^*) + \sum_{\substack{k=2 \\ k \text{ even}}}^{m-i-1} (x_{i+k-1} y_k^* + x_{i+k+1} y_k^* + x_{i+k} y_{k+1}^* + x_{i+k} y_{k-1}^*) \\ &= x_{i-1} y_0^* + x_{i+1} y_0^* + x_i y_1^* + \sum_{k=2}^{m-i} x_{i+k-1} y_k^* + \sum_{k=1}^{m-i-1} x_{i+k+1} y_k^* \\ &= -x_m y_{m-i+1}^* + \sum_{k=0}^{m-(i-1)} x_{i-1+k} y_k^* + \sum_{k=0}^{m-(i+1)} x_{i+1+k} y_k^*. \end{aligned}$$

Applying our definitions (4.1) and $\mathbb{Z}/2\mathbb{Z}$ coefficients, we have shown that

$$(4.2) \quad \partial u_i = \begin{cases} x_m y_{m-i+1}^* + u_{i-1} + u_{i+1} & i \text{ odd} \\ x_m y_{m-i+1}^* & i \text{ even.} \end{cases}$$

Second, we check $x_i y_0^* \rightarrow u_i$ respects the filtration. Let $0 < k < m - i$. By (2.2, 2.4) with i even,

$$\begin{aligned} fl(x_{i+k} y_k^*) &= \begin{cases} (fl(x_{i+k-1}) + (1, 0)) + (fl(y_{k-1}^*) - (1, 0)) & k \text{ odd} \\ (fl(x_{k-1}) - (0, a)) + (fl(y_{k-1}^*) + (0, a)) & k \text{ even} \end{cases} \\ &= fl(x_{i+k-1} y_{k-1}^*) \\ fl(x_m y_{m-i}^*) &= \begin{cases} (fl(x_{m-1}) - (0, a + c)) + (fl(y_{m-i-1}^*) + (0, a)) & i > 0 \text{ or } q > p \\ (fl(x_{m-1}) - (0, a + c)) + (fl(y_{m-i-1}^*) + (0, a + d)) & i = 0, q = p \end{cases} \\ &= \begin{cases} fl(x_{m-1} y_{m-i-1}^*) - (0, c) & i > 0 \text{ or } q > p \\ fl(x_{m-1} y_{m-i-1}^*) - (0, c - d) & i = 0, q = p. \end{cases} \end{aligned}$$

For i odd,

$$\begin{aligned} fl(x_{i+k} y_k^*) &= (fl(x_{i+k-2}) + (1, -a)) + (fl(y_{k-2}^*) + (-1, a)) \quad k \text{ even} \\ &= fl(x_{i+k-2} y_{k-2}^*) \quad k \text{ even.} \end{aligned}$$

We have therefore shown that

$$(4.3) \quad fl(x_i y_0^*) = \begin{cases} fl(x_{i+k} y_k^*) & i \text{ even or } (i \text{ odd, } k \text{ even}) \\ fl(x_m y_{m-i}^*) + (0, c) & i \text{ even, } (i > 0 \text{ or } q > p) \\ fl(x_m y_{m-i}^*) + (0, c - d) & i = 0, q = p. \end{cases}$$

We require $fl(x_i y_0^*) \geq fl(x_{i+k} y_k^*)$ for each k summed over in u_i . By (4.3), this is equivalent to the condition that $c \geq 0$ and that $d \leq c$ if $q = p$, which is true by hypothesis.

Now suppose that $c > 0$ and that $d < c$ if $q = p$. Letting $\tau = \tau(C \otimes D_r^*)$, note $fl(x_0 y_0^*) = (0, \tau)$, $fl(x_1 y_0^*) = (1, \tau)$, and $fl(x_2 y_0^*) = (1, \tau - a)$. In addition, it follows from (4.3) that

$$fl(x_m y_m^*) = \begin{cases} (0, \tau - c) & q > p \\ (0, \tau - (c - d)) & q = p \end{cases} \quad \text{and} \quad fl(x_m y_{m+1}^*) = \begin{cases} (-1, \tau - c) & q > p \\ (-(a + d), \tau - (c - d)) & q = p. \end{cases}$$

By (4.2), $\partial u_0 = x_m y_{m+1}^*$. Notice $x_m y_{m+1}^*$ has i -coordinate < 0 , while u_0 has $i = 0$; thus $\partial^{\text{vert}} u_0 = 0$. By (4.2), $\partial u_1 = x_m y_m^* + u_0 + u_2$. Notice $x_m y_m^*$ and u_2 have j -coordinate $< \tau$, while both u_0 and u_1 have $j = \tau$; thus $\partial^{\text{horz}} u_1 = u_0$. \square

It is easily seen that $\{u_i \mid i < m\} \cup \{x_m y_j \mid j \leq nr\}$ with its reflection in the above proof is a basis for $P_r \in [(1, a)_1^p, 0, c, ((-1, -a)_1^q, 0, -d)_1^r]$. Upon further basis changes, this set can be shown to belong to a simultaneously vertically and horizontally simplified basis for $(C \otimes D_r^*)$, from which one obtains $[C] - r[D_1] = [P_r]$, which we leave as an exercise for the dedicated reader.

5. FLOER COMPLEXES OF SELECTED L -SPACE KNOTS

In this section, we find formulas for the Alexander polynomials of certain cables of torus knots. The knots we consider are all L -space knots, and thus the computations of the Alexander polynomials in fact gives us the knot Floer complexes of these knots.

This first lemma involves taking a well-known formula for the Alexander polynomial of iterated torus knots and grouping the terms in order to simplify the expression, for example by noticing telescoping sums.

Lemma 5.1. *The Alexander polynomial of the $(m, mp(p-1)+1)$ -cable of the $(p, p+1)$ -torus knot is*

$$\Delta_{T_{p,p+1;m,mp(p-1)+1}}(t) = \sum_{i=0}^{mp(p-1)} t^{im} - t \left(\sum_{i=0}^{m-1} t^{i(mp^2-mp+1)} \right) \left(\sum_{j=0}^{p-2} t^{jmp} \left(\sum_{k=0}^j t^{km} + \sum_{k=j+1}^{p-1} t^{km-1} \right) \right).$$

Similarly, the Alexander polynomial of the $(m, mp(p-1)-1)$ -cable of the $(p, p+1)$ -torus knot is

$$\Delta_{T_{p,p+1;m,mp(p-1)-1}}(t) = -t \sum_{i=0}^{mp(p-1)-2} t^{im} + \left(\sum_{i=0}^{m-1} t^{i(mp^2-mp-1)} \right) \left(\sum_{j=0}^{p-2} t^{jmp} \left(\sum_{k=0}^j t^{km} + \sum_{k=j+1}^{p-1} t^{km+1} \right) \right).$$

Proof. We know that

$$\Delta_{T_{p,p+1;m,mp(p-1)+1}}(t) = \frac{(t^{mp(p+1)} - 1)(t^m - 1)}{(t^{mp} - 1)(t^{m(p+1)} - 1)} \cdot \frac{(t^{m(mp(p-1)+1)})(t - 1)}{(t^m - 1)(t^{mp(p-1)+1} - 1)}.$$

Let

$$P(t) = \sum_{i=0}^{mp(p-1)} t^{im} - t \left(\sum_{i=0}^{m-1} t^{i(mp^2-mp+1)} \right) \left(\sum_{j=0}^{p-2} t^{jmp} \left(\sum_{k=0}^j t^{km} + \sum_{k=j+1}^{p-1} t^{km-1} \right) \right).$$

To prove the lemma, we will show that

$$(5.1) \quad (t^{mp}-1)(t^{m(p+1)}-1)(t^m-1)(t^{mp(p-1)+1}-1) \cdot P(t) = (t^{mp(p+1)}-1)(t^m-1)(t^{m(mp(p-1)+1)})(t-1).$$

We first consider

$$(5.2) \quad (t^{mp}-1)(t^{m(p+1)}-1)(t^m-1)(t^{mp(p-1)+1}-1) \sum_{i=0}^{mp(p-1)} t^{im}.$$

Note the telescoping sum

$$(t^m-1) \sum_{i=0}^{mp(p-1)} t^{im} = t^{m^2 p^2 - m^2 p + m} - 1.$$

Hence (5.2) is equal to

$$(5.3) \quad (t^{mp}-1)(t^{m(p+1)}-1)(t^{mp(p-1)+1}-1)(t^{m^2 p^2 - m^2 p + m} - 1).$$

Next, we consider

$$(t^{mp}-1)(t^{m(p+1)}-1)(t^m-1) \cdot (t^{mp(p-1)+1}-1)(-t) \left(\sum_{i=0}^{m-1} t^{i(mp^2 - mp + 1)} \right) \left(\sum_{j=0}^{p-2} t^{jmp} \left(\sum_{k=0}^j t^{km} + \sum_{k=j+1}^{p-1} t^{km-1} \right) \right).$$

We first notice that

$$(5.4) \quad (t^{mp(p-1)+1}-1)(-t) \left(\sum_{i=0}^{m-1} t^{i(mp^2 - mp + 1)} \right) = -t \cdot (t^{m^2 p^2 - m^2 p + m} - 1).$$

Grouping the remaining factors yields

$$\begin{aligned} & (t^{mp}-1)(t^{m(p+1)}-1)(t^m-1) \left(\sum_{j=0}^{p-2} t^{jmp} \left(\sum_{k=0}^j t^{km} + \sum_{k=j+1}^{p-1} t^{km-1} \right) \right) \\ &= (t^{mp}-1)(t^{m(p+1)}-1) \left(\sum_{j=0}^{p-2} t^{jmp} (t^{jm+m} - 1 + t^{mp-1} - t^{jm+m-1}) \right) \\ &= (t^{mp}-1)(t^{m(p+1)}-1) \left(\sum_{j=0}^{p-2} t^{jmp} (t^{mp-1} - 1 + t^{jm+m-1}(t-1)) \right) \\ &= (t^{mp}-1)(t^{m(p+1)}-1) \left((t^{mp-1}-1) \sum_{j=0}^{p-2} t^{jmp} + t^{m-1}(t-1) \sum_{j=0}^{p-2} t^{jm(p+1)} \right) \\ (5.5) \quad &= (t^{m(p+1)}-1)(t^{mp-1}-1)(t^{mp(p-2)+mp}-1) + (t^{mp}-1)t^{m-1}(t-1)(t^{mp^2-m}-1). \end{aligned}$$

We have shown that the lefthand side of (5.1) is equal to

$$(5.3) + (5.4) \cdot (5.5),$$

where we have used telescoping sums to eliminate the summation from each of those expressions. It is straightforward to verify that this is equal to the righthand side of (5.1).

The calculation for

$$\Delta_{T_{p,p+1;m,mp(p-1)-1}}(t)$$

follows in a similar manner. □

The following lemma is essentially a restatement of Lemma 5.1 in our language for the ε -equivalence classes of staircase complexes, i.e., knot Floer complexes of L -space knots.

Lemma 5.2. *Let*

$$(5.6) \quad \begin{aligned} (x_s)_{s=1}^{2mp(p-1)} &= (((i, m-i)_1^j, (i-1, m-i+1)_1^{p-j})_{j=1}^{p-1})_{i=1}^m \\ (y_s)_{s=1}^{2mp(p-1)} &= (((m-i, i)_1^j, (m-i+1, i-1)_1^{p-j})_{j=1}^{p-1})_{i=1}^m. \end{aligned}$$

Then for positive p and m ,

$$[T_{p,p+1;m,(p-1)pm+1}] = [(x_s)_{s=1}^{mp(p-1)}] \quad \text{and} \quad [T_{p,p+1;m,(p-1)pm-1}] = [(y_s)_{s=2}^{mp(p-1)}].$$

Proof. We know that $CFK^\infty(T_{p,p+1;m,(p-1)pm\pm 1})$ are staircases. We translate backwards from the sequences of step lengths to the corresponding summations. Set $M = 2mp(p-1)$. Construct the polynomial $\sum_{i=0}^M (-1)^i t^{n_i}$ for which $n_s - n_{s-1} = x_s$. Setting $n_0 = 0$, note

$$\sum_{i=0, i \text{ even}}^M t^{n_i} = \sum_{i=0}^{mp(p-1)} t^{im},$$

which are all the terms with positive coefficient. Setting $n_1 = 1$, note

$$\sum_{i=1, i \text{ odd}}^{M-1} (-t^{n_i}) = -t \sum_{i=0}^{m-1} \sum_{j=0}^{p-2} t^{(i(p-1)+j)pm} \left(\sum_{k=0}^j t^{km} t^i + \sum_{k=j+1}^{p-1} t^{km} t^{i-1} \right),$$

which are all the terms with negative coefficient. Putting these together and rearranging slightly gives the first polynomial in the statement of Lemma 5.1. Now construct $\sum_{i=0}^M (-1)^i t^{n_i}$ for which $n_s - n_{s-1} = y_s$. The positive coefficients are the same, and setting $n_1 = m-1$, note

$$\sum_{i=1, i \text{ odd}}^{M-1} (-t^{n_i}) = -t^{-1} \sum_{i=0}^{m-1} \sum_{j=0}^{p-2} t^{(i(p-1)+j)pm} \left(\sum_{k=0}^j t^{km} t^{m-i} + \sum_{k=j+1}^{p-1} t^{km} t^{m-i+1} \right).$$

$(y_s)_{s=2}^{2mp(p-1)-1}$ corresponds to $\sum_{i=1}^{M-1} (-1)^i t^{n_i}$, so we remove t^{n_0} and t^{n_M} , which are the first and last terms of the positive summation, and multiply through by $-t^{-m+1}$ to renormalize. Rearranging slightly gives the second polynomial in the statement of Lemma 5.1. \square

The corollary below follows by taking $m = 1$ (cf. [HLR10, Proposition 6.1]).

Corollary 5.3. *Let*

$$(5.7) \quad (t_s)_{s=1}^{2(p-1)} = (j, p-j)_{j=1}^{p-1}.$$

Then for positive p ,

$$[T_{p,p+1}] = [(t_s)_{s=1}^{p-1}].$$

Proof. Take $m = 1$ in (x_s) , since $T_{p,p+1;1,(p-1)p+1} = T_{p,p+1}$. Then $[(x_s)_{s=1}^{2mp(p-1)}]$ simplifies as

$$[(x_s)_{s=1}^{2mp(p-1)}] = [(((i, 1-i)_1^j, (i-1, 1-i+1)_1^{p-j})_{j=1}^{p-1})_{i=1}^1] = [((1, 0)_1^j, (0, 1)_1^{p-j})_{j=1}^{p-1}] = [(j, p-j)_{j=1}^{p-1}].$$

It follows that $[(x_s)_{s=1}^{mp(p-1)}] = [(t_s)_{s=1}^{p-1}]$. \square

Note that (x_s) and (y_s) have 0 entries where the terms with positive coefficient cancel those with negative coefficient (see proof of Lemma 5.2). This occurs when $i = 1$ or m in (5.6). We can therefore simplify the complexes $[(x_s)_{s=1}^{2p(p-1)}]$ and $[(y_s)_{s=1}^{2p(p-1)+1}]$ as follows:

$$\begin{aligned}
[(x_s)_{s=1}^{2p(p-1)}] &= [((1, m-1)_1^j, (0, m)_1^{p-j})_{j=1}^{p-1}] \\
&= [((1, m-1)_1^{j-1}, 1, m-1, 0, (p-j)m)_{j=1}^{p-1}] \\
&= [((1, m-1)_1^{j-1}, 1, (p-j+1)m-1)_{j=1}^{p-1}] \\
[(y_s)_{s=1}^{2p(p-1)+1}] &= [((m-1, 1)_1^j, (m, 0)_1^{p-j})_{j=1}^{p-1}, m-2] \\
&= [(m-1, (1, m-1)_1^{j-1}, 1, (p-j)m, 0)_{j=1}^{p-1}, m-1, 0, -1] \\
&= [m-1, ((1, m-1)_1^{j-1}, 1, (p-j+1)m-1)_{j=1}^{p-1}, 0, -1].
\end{aligned}$$

Letting $j \rightarrow j+1$, we can write

$$\begin{aligned}
(5.8) \quad [(x_s)_{s=1}^{2p(p-1)}] &= [((1, m-1)_1^j, 1, (p-j)m-1)_{j=0}^{p-2}] \\
[(y_s)_{s=2}^{2p(p-1)+1}] &= [((1, m-1)_1^j, 1, (p-j)m-1)_{j=0}^{p-3}, (1, m-1)_1^{p-2}, 1, 2m-2].
\end{aligned}$$

6. ARCHIMEDEAN EQUIVALENCE CLASSES OF \mathcal{F}

We can now combine the results of Sections 3, 4, and 5 to find new Archimedean equivalence classes in \mathcal{F} . In the following lemma, we define the knots that will be at the heart of Theorem 1.

Lemma 6.1. *Set $m = i + 1$ and $p = |j| + 3$ for $i, j \in \mathbb{Z}$. Define*

$$K_{(i,j)} = \begin{cases} T_{p,p+1;m,(p-1)pm+1} \# -T_{pm,pm+1} & i > 0, j \geq 0 \\ T_{p,p+1;m,(p-1)pm+1} \# -T_{p,p+1;m,(p-1)pm-1} & i > 0, j < 0 \\ T_{p,p+1} \# -T_{2,3;[p/2],2[p/2]+1} & i = 0, j \geq 0. \end{cases}$$

Then $\llbracket K_{(0,0)} \rrbracket = 0$ and for $(i, j) > (0, 0)$,

$$\llbracket K_{(i,j)} \rrbracket \sim_{\text{Ar}} \begin{cases} [1, i, 1, 2i+1+j(i+1)] & j \geq 0 \\ [(1, i)_1^{-j}, 1, i, 1, 2i+1] & j \leq 0. \end{cases}$$

Proof. Making inductive use of Lemma 3.1 on $[(t_s)_{s=1}^{p-1}]$ of Corollary 5.3,

$$(6.1) \quad \llbracket T_{p,p+1} \rrbracket = \sum_{k=1}^n [k, p-k] + [(t_s)_{s=2n+1}^{p-1}]$$

for any $0 \leq n < [p/2]$. Applying Lemma 3.1 to $[(x_s)_{s=1}^{mp(p-1)}]$ and $[(y_s)_{s=2}^{mp(p-1)}]$ of Lemma 5.2 gives

$$\begin{aligned}
(6.2) \quad \llbracket T_{p,p+1;m,(p-1)pm+1} \rrbracket &= [(x_s)_{s=1}^{2p(p-1)}] + [(x_s)_{s=2p(p-1)+1}^{mp(p-1)}] \\
\llbracket T_{p,p+1;m,(p-1)pm-1} \rrbracket &= [(y_s)_{s=2}^{2p(p-1)+1}] + [(y_s)_{s=2p(p-1)+2}^{mp(p-1)}].
\end{aligned}$$

Making inductive use of Lemma 3.2 on (5.8), we obtain

$$\begin{aligned}
(6.3) \quad [(x_s)_{s=1}^{2p(p-1)}] &= \sum_{k=0}^{p-2} [(1, m-1)_1^k, 1, (p-k)m-1] \\
[(y_s)_{s=2}^{2p(p-1)+1}] &= \sum_{k=0}^{p-3} [(1, m-1)_1^k, 1, (p-k)m-1] + [(1, m-1)_1^{p-2}, 1, 2m-2].
\end{aligned}$$

We use the class decompositions (6.1, 6.2, 6.3) along with the sequence definitions (5.6, 5.7) in the following calculations.

Case 1: $i > 0, j \geq 0$. We have $\llbracket K_{(i,j)} \rrbracket = \llbracket T_{p,p+1;m,(p-1)pm+1} \rrbracket - \llbracket T_{pm,pm+1} \rrbracket$. Hence $\llbracket K_{(i,j)} \rrbracket$ is given by

$$\begin{aligned} & \sum_{k=0}^{p-2} [(1, m-1)_1^k, 1, (p-k)m-1] + [(x_s)_{s=2p(p-1)+1}^{mp(p-1)}] - [1, pm-1] - [(t_s)_{s=3}^{pm-1}] \\ &= \sum_{k=1}^{p-2} [(1, m-1)_1^k, 1, (p-k)m-1] + [2, (x_s)_{s=2p(p-1)+2}^{mp(p-1)}] - [2, (t_s)_{s=4}^{pm-1}], \end{aligned}$$

where the $[1, pm-1]$ term cancels. By Lemmas 4.1 and 4.2, $[1, m-1, 1, (p-1)m-1] \gg$ the other remaining terms so that $\llbracket K_{(i,j)} \rrbracket \sim_{\text{Ar}} [1, m-1, 1, (p-1)m-1]$.

Case 2: $i > 0, j < 0$. We have $\llbracket K_{(i,j)} \rrbracket = \llbracket T_{p,p+1;m,(p-1)pm+1} \rrbracket - \llbracket T_{p,p+1;m,(p-1)pm-1} \rrbracket$. Hence $\llbracket K_{(1,j)} \rrbracket = [(1, 1)_1^{p-2}, 1, 3] - [(1, 1)_1^{p-2}, 1, 2] \sim_{\text{Ar}} [(1, 1)_1^{p-2}, 1, 3]$, and for $i \geq 2$, $\llbracket K_{(i,j)} \rrbracket$ is given by

$$\begin{aligned} & \sum_{k=0}^{p-2} [(1, m-1)_1^k, 1, (p-k)m-1] + [(x_s)_{s=2p(p-1)+1}^{mp(p-1)}] - \sum_{k=0}^{p-3} [(1, m-1)_1^k, 1, (p-k)m-1] \\ & \quad - [(1, m-1)_1^{p-2}, 1, 2m-2] - [(y_s)_{s=2p(p-1)+2}^{mp(p-1)}] \\ &= [(1, m-1)_1^{p-2}, 1, 2m-1] - [(1, m-1)_1^{p-2}, 1, 2m-2] + [2, (x_s)_{s=2p(p-1)+2}^{mp(p-1)}] - [2, (y_s)_{s=2p(p-1)+3}^{mp(p-1)}], \end{aligned}$$

where each $[(1, m-1)_1^k, 1, (p-k)m-1]$ term for $0 \leq k \leq p-3$ cancels. By Lemmas 4.1 and 4.2, $[(1, m-1)_1^{p-2}, 1, 2m-1] \gg$ the other remaining terms so that $\llbracket K_{(i,j)} \rrbracket \sim_{\text{Ar}} [(1, m-1)_1^{p-2}, 1, 2m-1]$.

Case 3: $i = 0, j > 0$. Letting $q = \lceil p/2 \rceil$, we can rephrase the lemma for this case as

$$K_{(0, 2q - \frac{7 \pm 1}{2})} = T_{2q, 2q \pm 1} \# -T_{2, 3; q, 2q \pm 1} \implies \llbracket K_{(0,j)} \rrbracket \sim_{\text{Ar}} [1, 0, 1, 1+j] = [2, 1+j].$$

For j odd, we have $\llbracket K_{(0,j)} \rrbracket = \llbracket T_{2q, 2q+1} \rrbracket - \llbracket T_{2, 3; q, 2q+1} \rrbracket$. Hence $\llbracket K_{(0,1)} \rrbracket = [1, 3] + [2] - [1, 3] = [2] \sim_{\text{Ar}} 2[2] = [2, 2]$, and for $j \geq 3$, $\llbracket K_{(0,j)} \rrbracket$ is given by

$$\begin{aligned} & \sum_{k=1}^2 [k, 2q-k] + [(t_s)_{s=5}^{2q-1}] - [1, 2q-1] - [(x_s)_{s=5}^{2q}] \\ &= [2, 2q-2] + [3, (t_s)_{s=6}^{2q-1}] - [2, q-2, (x_s)_{s=7}^{2q}], \end{aligned}$$

where the $[1, 2q-1]$ term cancels. By Lemma 4.1, $[2, 2q-2] \gg$ the other remaining terms so that $\llbracket K_{(0,j)} \rrbracket \sim_{\text{Ar}} [2, 2q-2] = [2, p-2]$. For j even, we similarly find $\llbracket K_{(0,j)} \rrbracket \sim_{\text{Ar}} [2, 2q-3] = [2, p-2]$.

The result for $(i, j) > (0, 0)$ follows by substituting $(i+1)$ for m and $(|j|+3)$ for p in each case, and $\llbracket K_{(0,0)} \rrbracket = \llbracket T_{3,4} \rrbracket - \llbracket T_{2,3;2,3} \rrbracket = [1, 2] - [1, 2] = 0$. \square

We now conclude with the proof of the main theorem.

Proof of Theorem 1. For $i < i'$, we have that $\llbracket K_{(i,j)} \rrbracket \ll \llbracket K_{(i',j')} \rrbracket$ by Lemma 4.1. For $j < j'$, we have that $\llbracket K_{(i,j)} \rrbracket \ll \llbracket K_{(i,j')} \rrbracket$ by Lemma 4.1 ($i = 0$) and Lemma 4.2 ($i > 0$). Thus,

$$(i, j) < (i', j') \implies \llbracket K_{(i,j)} \rrbracket \ll \llbracket K_{(i',j')} \rrbracket.$$

It follows that $\{H_{\llbracket K_{(i,j)} \rrbracket} \mid (i, j) \in S\}$ is a filtration on \mathcal{F} with

$$H_{\llbracket K_{(i,j)} \rrbracket} \subset H_{\llbracket K_{(i',j')} \rrbracket} \quad \text{if } (i, j) < (i', j')$$

and $\mathbb{Z} \subset H_{\llbracket K_{(i',j')} \rrbracket} / H_{\llbracket K_{(i,j)} \rrbracket}$, generated by $\llbracket K_{(i',j')} \rrbracket$. Recall the map $\phi : \mathcal{C} \rightarrow \mathcal{F}$. Letting

$$\mathcal{F}_{(i,j)} = \phi^{-1}[H_{\llbracket K_{(i,j)} \rrbracket}],$$

we pull back to a filtration $\{\mathcal{F}_{(i,j)} \mid (i,j) \in S\}$ on \mathcal{C} with $\mathbb{Z} \subset \mathcal{F}_{(i',j')}/\mathcal{F}_{(i,j)}$ for $(i,j) < (i',j')$. \square

REFERENCES

- [CHH12] Tim D. Cochran, Shelly Harvey, and Peter Horn, *Filtering smooth concordance classes of topologically slice knots*, preprint, 2012.
- [CHL09] Tim D. Cochran, Shelly Harvey, and Constance Leidy, *Knot concordance and higher-order Blanchfield duality*, *Geom. Topol.* **13** (2009), no. 3, 1419–1482.
- [COT03] Tim D. Cochran, Kent E. Orr, and Peter Teichner, *Knot concordance, Whitney towers and L^2 -signatures*, *Ann. of Math. (2)* **157** (2003), no. 2, 433–519.
- [Hed09] Matthew Hedden, *On knot Floer homology and cabling II*, *Int. Math. Res. Not. IMRN* (2009), no. 12, 2248–2274.
- [HLR10] Matthew Hedden, Charles Livingston, and Daniel Ruberman, *Topologically slice knots with nontrivial Alexander polynomial*, preprint.
- [Hom11] Jennifer Hom, *The knot Floer complex and the smooth concordance group*, preprint (2011), arXiv:1111.6635v1.
- [Hom12] ———, *Bordered Heegaard Floer homology and the tau-invariant of cable knots*, preprint (2012), arXiv:1202.1463v1.
- [LOT08] Robert Lipshitz, Peter Ozsváth, and Dylan Thurston, *Bordered Heegaard Floer homology: Invariance and pairing*, preprint (2008), arXiv:0810.0687v4.
- [OS03] Peter Ozsváth and Zoltán Szabó, *Knot Floer homology and the four-ball genus*, *Geom. Topol.* **7** (2003), 615–639.
- [OS04] ———, *Holomorphic disks and knot invariants*, *Adv. Math.* **186** (2004), no. 1, 58–116.
- [OS05] ———, *On knot Floer homology and lens space surgeries*, *Topology* **44** (2005), no. 6, 1281–1300.
- [Ras03] Jacob Rasmussen, *Floer homology and knot complements*, Ph.D. thesis, Harvard University, 2003.
- [Ras04] ———, *Lens space surgeries and a conjecture of Goda and Teragaito*, *Geom. Topol.* **8** (2004), 1013–1031.

DEPARTMENT OF PHYSICS, UNIVERSITY OF CALIFORNIA, SAN DIEGO, LA JOLLA, CA 92093

E-mail address: shancock@physics.ucsd.edu

DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY, NEW YORK, NY 10027

E-mail address: hom@math.columbia.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MI 48109

E-mail address: mnewman@umich.edu